Partial Differential Equations

(Semester II; Academic Year 2024-25)

Indian Statistical Institute, Bangalore

Final Exam

Duration: 3 hrs — Maximum Marks: 50

1. Consider the IVP $u_x^2 + u_y^2 = 1$, $u(x, y) = 0$ on the line $x + y = 1$.
   (a) Discuss the existence and uniqueness of the IVP. (5)
   (b) Solve the IVP (5)

2. Consider the PDE $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$ where λ is a scalar and Ω is a bounded open set. If $\lambda \leq 0$, prove that $u \equiv 0$. (5)

3. Suppose u solves: (5)
$$\begin{cases} u_t - \Delta u = u \text{ in } \Omega \times (0, T), \\ u(x, 0) = 0 \text{ in } \Omega, \\ u(x, t) = 0 \text{ on } \partial\Omega \times [0, T]. \end{cases}$$
   Then show that $u(x, t) = 0$ in $\Omega \times (0, T)$.

4. Show that if u satisfies the heat equation $u_t - \Delta u = 0$ in $\Omega \times (0, T)$, then the following maximum principle holds: (5)
$$\sup_{\Omega \times (0,T)} u(x, t) = \sup_{\Gamma_T} u(x, t).$$

5. (a) Prove the characteristic parallelogram property for one dimensional wave equations. (5)
   (b) Use characteristic parallelogram property to solve (5)
$$\begin{cases} u_{tt} - u_{xx} = 0, \quad x > 0, \quad t > 0, \\ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \\ u(0, t) = h(t). \end{cases}$$

6. Integrate the wave equation $u_{tt} - c^2 u_{xx} = f(x, t)$ in the characteristic triangle $P(x, t)$, $Q(x - ct, 0)$, $R(x + ct, 0)$ to derive a formula for the solution. (5)

7. Is there an f in $L^1(\mathbb{R})$ such that $f * f = f$ and $f \neq 0$? (5)

8. Find a smooth function $a : \mathbb{R}^2 \to \mathbb{R}$ such that, for the equation of the form (5)
$$a(x, y)\, u_x + u_y = 0,$$
   there does not exist any solution in the entire $\mathbb{R}^2$ for any nonconstant initial value prescribed on $\{y = 0\}$.

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